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NOTES

Edited by **Vadim Ponomarenko**

Why Do All Triangles Form a Triangle?

Ian Stewart

Abstract. We provide a geometric explanation, based on symmetry, for why the moduli space of all triangles up to similarity is itself a triangle. Symmetries occur because the lengths of the sides define triples in $\mathbb{R}^3$ so are acted on by the symmetric group $\mathbb{S}_3$, which is isomorphic to the symmetry group $\mathbb{D}_3$ of an equilateral triangle. The moduli space for triangles is a fundamental domain for the action of $\mathbb{D}_3$ on an equilateral triangle in $\mathbb{R}^3$ determined by all triangles with unit perimeter and is chosen from a subdivision into six congruent triangles. Isosceles and equilateral triangles occupy special locations determined by their symmetries. The sides of a right triangle lie on one of three double cones in $\mathbb{R}^3$, and those of unit perimeter lie on a segment of a hyperbola in the moduli space.

Gaspar and Neto [**1**] prove an elegant result, whose essence, in slightly different terminology, is that the moduli space of all triangles, equivalent up to similarity, is itself a triangle.

In general, a moduli space is a geometric space whose points represent equivalence classes of geometric objects while preserving important features of the relationships between those objects. Here, the equivalence relation is "the same up to a rigid motion and a similarity," and the moduli space is the orbit space of a transformation group. Its natural topology preserves continuity in the sense that small changes in the sides of the triangle lead to small changes in the location of the corresponding point in the moduli space. Symmetry properties of triangles are also preserved. So the moduli space captures many features of the set of all triangles in a natural way. See Behrend [**2**] for a more advanced treatment of moduli spaces, which, among other things, includes the results in this note and extensions to other spaces of triangles. The aim here is to give a simple, self-contained introduction to one accessible example of these ideas.

Gaspar and Neto represent a triangle by a triple $(a, b, c) \in \mathbb{R}^3$ defined by its three sides and normalize its size by requiring

$$a + b + c = 1. \tag{1}$$

They add two further systems of inequalities to define a genuine triangle and remove any ambiguity in the ordering of the sides:

$$a \le b + c, \quad b \le a + c, \quad c \le a + b, \tag{2}$$

$$0 \le a \le b \le c. \tag{3}$$

Degenerate triangles, with some side or angle equal to 0, are permitted, since these aid the analysis. They then substitute $c = 1 - a - b$ to transform the above conditions

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into a simpler form and analyze the result in the (a, b)-plane using coordinate geometry. This leads to a triangular region that not only classifies all triangles uniquely up to similarity but also locates equilateral, isosceles, right, acute, and obtuse triangles relative to each other.

The idea is interesting and the result intriguing, but the proof is a calculation that sheds little light on *why* the moduli space is a triangle. It must be determined by some subset of the plane (1), but is there a structural reason for the triangular shape? The answer is "yes," the cause is symmetry, and the proof is geometric.

There are three ways to approach problems with symmetry:

(1) Factor out the symmetry early on, and work with a "reduced" problem.

The other two approaches remove the symmetry only when it becomes an obstacle to further progress. Abstractly, if a group Γ acts on a space X, we can either

(2) Pick representatives of the Γ-orbits and work with those, or

(3) Work with the orbit space X/Γ. This case subdivides further into two alternatives: Consider the orbit space as such, or work on X itself but thinking "modulo Γ."

All three approaches have advantages and disadvantages, and in practice, it is common to switch between them as required.

A simple analogy is arithmetic modulo n. For calculations, it is often best to identify $\mathbb{Z}_n$ with $\{0, 1, 2, \ldots, n-1\}$. For theoretical purposes, it is best to consider elements of $\mathbb{Z}_n$ as equivalence classes of $\mathbb{Z}$ modulo congruence $\equiv \pmod{n}$. In practice, we often apply the third method: Calculate in $\mathbb{Z}$, while remembering, when necessary, that we can throw away multiples of n. There is then no need to reduce everything back to the interval $[0, n-1]$ at every step of the calculation.

The symmetry group we need here is the group $\mathbb{S}_3$ of permutations of the three sides a, b, c. Equations (1) and (2) respect the symmetry, but (3) removes it by normalizing the order. To retain the symmetry, we replace (3) by

$$a \geq 0, \quad b \geq 0, \quad c \geq 0. \tag{4}$$

The disadvantage of doing this is that each triangle now appears up to six times (three for isosceles triangles and one for equilateral). The advantage is that we can now use symmetry to analyze the geometry of these triples.

Start with $\mathbb{R}^3$, with coordinates (a, b, c) instead of the usual (x, y, z), to retain the traditional notation for triangles in which the sides are denoted a, b, c. Equation (1) defines a plane P. It is orthogonal to the main diagonal, the line passing through $(0, 0, 0)$ and $(1, 1, 1)$, and meets the main diagonal at $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Inequalities (4) determine the positive octant of $\mathbb{R}^3$, bounded by the three planes $a = 0, b = 0, c = 0$. It is clear that the intersection of P with the positive octant is an equilateral triangle Δ with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. This is the large triangle in Figure 1, which views the plane P from the direction of the main diagonal. The projections of the axes are three lines at angles of $2\pi/3$.

The symmetric group $\mathbb{S}_3$ acts (orthogonally) on $\mathbb{R}^3$ by permuting coordinates, and this induces an action on Δ. As is well known, geometrically this induces the symmetry group of rigid motions of Δ, the dihedral group $\mathbb{D}_3$, which is isomorphic to $\mathbb{S}_3$.

The region defined by the triangle inequalities (2) is the smaller equilateral triangle Ω, drawn with a thicker line, with vertices at $(\frac{1}{2}, \frac{1}{2}, 0)$, $(0, \frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, 0, \frac{1}{2})$. The region Ω corresponds to (possibly degenerate) triangles, up to permutation of their sides. This region is also symmetric under the action of $\mathbb{D}_3$.

Degenerate triangles occupy the boundary of Ω, where $a = b + c$, $b = a + c$, or $c = a + b$. Isosceles triangles lie on the axes of reflectional symmetry of Ω since there $a = b$, $b = c$, or $c = a$. The unique equilateral triangle of perimeter 1 lies at the center

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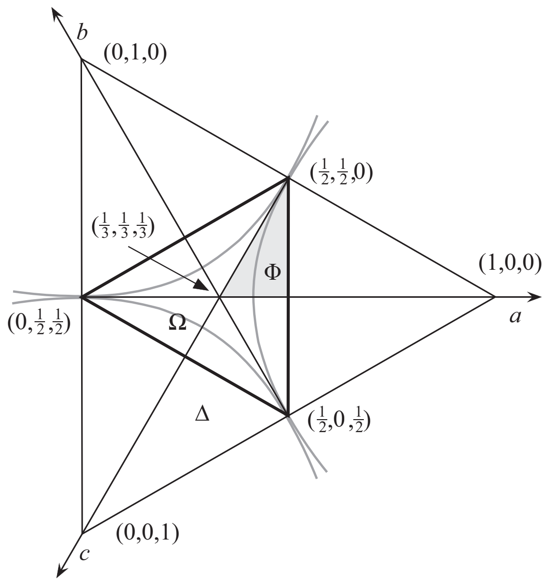

Figure 1. Geometry of the moduli space Φ for all triangles (shaded).

where $a = b = c = \frac{1}{3}$. These types of triangles can be defined by symmetries in the Euclidean plane and correspond to fixed-point spaces of subgroups of $\mathbb{D}_3$: the three axes (for subgroups of order 2) and the central point (the whole of $\mathbb{D}_3$). All other triangles have only trivial symmetry (for the subgroup of order 1).

At this stage, we can factor out the symmetry and remove the ambiguity created by permutations of the sides by passing to the orbit space of $\mathbb{D}_3$ on Ω. The civilized way to define this is to find a *fundamental domain*, a connected region of the plane that contains exactly one representative of each orbit.

The shaded region Φ is a natural choice of fundamental domain, one of six mutually congruent triangles that together give the whole of Ω. They map to each other by repeated reflection in the two sides that pass through the center. The moduli space of all triangles up to similarity therefore corresponds to Φ, which by construction is a triangle; indeed, a right triangle with angles $\pi/2, \pi/3, \pi/6$.

Right triangles are not determined by symmetry under rigid motions, and a further geometric argument is needed to locate them in the moduli space. There is some interesting two- and three-dimensional geometry here, and a class could learn a lot of geometry by filling in the details; I'll be brief and take the most direct route. Initially, ignore condition (1). Then a right triangle satisfies $a^2 + b^2 = c^2$ or a permutation $a^2 + c^2 = b^2$, $b^2 + c^2 = a^2$. Each equation defines a right circular double cone in $\mathbb{R}^3$, whose axis is, respectively, the c-axis, b-axis, a-axis. To visualize these, inscribe a circle on each square face of a cube whose faces are parallel to the axes of $\mathbb{R}^3$, and whose center is the origin. Use these as the bases of cones, with vertex the center of the cube. Cones are tangent to each other along radial lines from the origin to the midpoints of the edges of the cube, where the circles touch.

Right triangles in the diagram occur when the plane P meets one of these cones. So they lie on three conic sections. The conic sections turn out to be hyperbolas, which are mutually tangent at the vertices of Ω. They are drawn as gray curves. (The equation $a + b - ab = \frac{1}{2}$ given by Gaspar and Neto [**1**] can be rearranged as $(a - 1)(b - 1) = -\frac{1}{2}$, which is a rectangular hyperbola. The geometry explains why and provides a cone for it to be a section of.) Hyperbolas have two branches, but the other branch

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does not affect the picture. By continuity, acute triangles lie on the same side of the hyperbola as the center of the diagram, and obtuse ones lie on the other side, as Gaspar and Neto invite readers to find out.

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A New Proof of the Finsler–Hadwiger Inequality

Let a, b, c denote the side lengths, s the semiperimeter, r the inradius, R the circumradius, and S the area of triangle ABC. The well-known Finsler–Hadwiger inequality reads as

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}S + (a-b)^2 + (b-c)^2 + (c-a)^2.$$

We present what we believe to be a new and elementary proof for the above inequality. For other proofs, see [**1, 2, 3, 4**]. The Finsler–Hadwiger inequality can be rewritten as

$$a(s-a) + b(s-b) + c(s-c) \geq 2\sqrt{3}S.$$

By the arithmetic-geometric mean inequality and Heron's formula, we have

$$a(s-a) + b(s-b) + c(s-c) \geq 3\sqrt[3]{abc(s-a)(s-b)(s-c)} = 3S\sqrt[3]{\frac{4R}{s}}.$$

By the concavity of the sine function we have

$$\sin A + \sin B + \sin C \leq 3\sin\left(\frac{A+B+C}{3}\right) = 3\sin\left(\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2}.$$

By the sine law, we derive $a + b + c \leq 3\sqrt{3}R$ or equivalently $s \leq \frac{3\sqrt{3}}{2}R$. Therefore,

$$a(s-a) + b(s-b) + c(s-c) \geq 3S\sqrt[3]{\frac{8\sqrt{3}}{9}} = 2\sqrt{3}S.$$

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